cremental indentation step. This fact has been stated in the first paragraph of the introduction of Ref. 2 with a typical stress-strain curve for material with strain hardening and the implications that arise from the relative amount of incremental loading and the slope of the stress-strain curve, therefore, the degree of positive definiteness of the stiffness matrix obtained has been explained with reference to Fig. 2; and one method of diagnosis for ill behavior of the stiffness matrix has been explained with reference to Fig. 3 on page 1827 of Ref. 2.

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A Further Note on Laminar Incipient Separation

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Nomenclature

H = boundary-layer form factor
K = pressure gradient parameter
M = Mach number
S_w = total enthalpy function
T = temperature
τ = shear stress
θ_i = incipient compression surface deflection angle
$\bar{\chi}$ = viscous interaction parameter

Subscripts

o = flat plate, i.e., $K_o = 0$
s = separation, i.e., $\tau_w = 0$
tr = transformed
w = wall

REFERENCE 1 presented the effect of wall temperature on the incipient deflection angle, θ_i, i.e., that angle which the laminar boundary layer can negotiate without separating. The wall temperature was shown to enter into the valuation of λ in the equation

$$M\theta_i = \lambda\bar{\chi}^{1/2} \tag{1}$$

where

$$\lambda = -2.45K_s\,(-\Delta H_{tr}/\Delta K)^{1/2} \tag{2}$$

Although λ may be evaluated accurately through Eq. (2) by making use of the similar solutions of Ref. 2, its physical significance was not apparent.

Recalling that

$$\Delta H_{tr} \equiv H_{tr_s} - H_{tr_o}$$
$$\Delta K \equiv K_0 - K_s = -K_s,\ K_o = 0$$

and noting that, very accurately, independent of the wall-to-stagnation temperature ratio,

$$K_sH_{tr_s} \cong -0.275,\ -1 \leq S_w \leq 1$$

one obtains, upon substitution into Eq. (2),

$$\lambda = 1.288(1 - H_{tr_o}/H_{tr_s})^{1/2} \tag{3}$$

Therefore, λ is a function of the ratio of the transformed form factor at the beginning of the interaction and at the point of incipient separation. Finally, it is noted that the ratio H_{tr_o}/H_{tr_s} increases nonlinearly with S_w only through H_{tr_s} since $H_{tr_o} = 2.591\ (S_w + 1)$, H_{tr_s} being inversely proportional to the pressure gradient at separation.

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Comment on "Exact Solution of Certain Problems by Finite Element Method"

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IN his recent Note, Tong[1] proved that when the exact solution to the homogeneous Euler Equations of a positive definite functional[2] with one independent variable is known, and it is used as interpolation functions in the variational formulation of the finite element equations, the generalized displacements which are the solution to these equations constitute the exact solution to the problem at the nodal points regardless of the number or size of the elements used in the discretization. This property is often used in many one-dimensional problems and also in two-dimensions when a separation of variables is applicable. Typical problems of this kind are those concerning continuous beams, frame structures, some plate problems, and axisymmetric shells of revolution.

It is of interest to note, however, that an alternate derivation of the finite element equations has been used by engineers before the advent of the finite element method as such. If the generalized displacements are defined as the displacements at the element ends, the Euler Equations of the appropriate functional are equilibrium equations at the same points. In Tong's Note these are given in matrix form in Eq. (8). Obviously the entries of the stiffness matrix **K** are the end forces corresponding to unit values of the end displacements, and they can be obtained from the general solution to the homogeneous equations. Thus, when these end forces are obtained from the exact general solution, they are exact too.

Then, it remains to prove that the generalized forces in Eq. (8), **Q**, are also exact. The generalized forces are defined

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as minus the end restraint forces due to the loads acting in the element, when the ends are held fixed. The equivalence between this definition and that given by Tong from the variation of Eq. (2) can easily be shown by means of the virtual work equation where the virtual displacements are taken as the actual displacements in the element when one of the end displacements, q_i, is equal to one and all others are zero:

$$\frac{\partial}{\partial q_i}\left[\int_{x_i}^{x_{i+1}} pwdx\right] = \int_{x_i}^{x_{i+1}} p\delta w\, dx = \int_{x_i}^{x_{i+1}} G\delta wdx + Q_i$$

where G is the vector of internal forces associated with the degrees of freedom of the problem (a function of x) due to the element loads for fixed end condition, and Q_i is the generalized force associated with q_i. Since G constitutes a state of stress that corresponds to a fixed end condition, it immediately follows that the first term in the right-hand side of the preceding equation vanishes.

The validity of Eq. (8) can also be proved without reference to variational considerations by pointing out that the internal forces G satisfy equilibrium and compatibility inside each element and that the interelement compatibility and displacement boundary conditions are satisfied by definition of the generalized displacements; thus it only remains to insure equilibrium at the element ends and to satisfy the force boundary conditions that may be prescribed, which is done through Eq. (8). Finally, it may be concluded in agreement with Tong, that the solution of Eq. (8) for the nodal displacements is exact because both **K** and **Q** are exact.

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Comment on "Heat-Transfer Characteristies of Hot-Gas Ignition"

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Nomenclature

D_i = igniter nozzle throat diameter
D_p = test-section diameter
p_0 = stagnation pressure upstream nozzle
p_w = static pressure in the dead-air region
$\bar{p}_w$ = p_w/p_0
x = coordinate along test section
$\bar{x}$ = x/D_p

When investigating heat transfer during head-end hot-gas ignition, the authors[1] distinguished only two cases of flow pattern (Figs. 1a and 1d). One case corresponds to the reattachment of supersonic jet (Fig. 1a) and the other one has no reattachment at all (Fig. 1d). This is an over-simplification because there are two other intermediate cases.[2] One is significant for subsonic reattachment (Fig. 1c) and the other for mixed reattachment and oscillating flow (Fig. 1b). The type of flow appearing depends on the level of nondimensional pressure $\bar{p}_w$ in the dead-air region; that depends, however, on over-all pressure ratio and nozzle area to test-section area ratio. The heat transfer might be influenced very significantly by the different location or oscillating motion of the reattachment region. In my experiments the

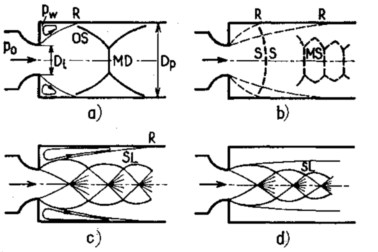

Fig. 1 Jet expansion from sonic convergent nozzles; R = reattachment line, OS = oblique shock, MD = Mach disk, SS = single shock, MS = multishock, SL = subsonic layer.

subsonic reattachment line defined by oil-film technique was located much further from nozzle $\bar{x} = 1.8$ for $D_p/D_i = 1.58$ than the supersonic one, $\bar{x} = 0.33$. The cyclic oscillation of the reattachment region is associated with strong pressure and shock-wave cyclic oscillation. The change of flow pattern is also cyclic, and therefore the wall is being touched by the supersonic and subsonic stream in turn. The oscillation is self-excited, and for each value of D_p/D_i it appears in a definite range of $\bar{p}_w$. These and some other features of the oscillating flow are described in detail in Refs. 2 and 3. It is therefore of interest to know how exact was the coincidence of reattachment line location with the heat-transfer maximum and to what types of reattachment the results given in Figs. 8–10 correspond.

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Reply by Author to W. M. Jungowski

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In answer to Professor Jungowski's first question, the point of reattachment was not determined during the experimental studies. However, the expanding jet correlations of Love et al. (Ref. 10 of the paper) were used to compute the point where a constant pressure jet boundary would intersect the wall. Of course, the actual point of reattachment is located somewhere in the shear layer encompassing this jet boundary. Also, the jet boundary predictions are based on the existence of constant pressure when, in actuality, the pressure rises as the shear layer approaches the wall and recompression begins. Nevertheless, it has been found that the predicted point of intersection of the constant pressure jet boundary with the wall does coincide with the point of maximum heat transfer to within the scale of the measurements (spacing between thermocouples was $\frac{1}{4}$ of a duct diameter).

The flow cases suggested by Professor Jungowski appear to be reasonable. Perhaps the transition from one flow regime to another furnishes an explanation of the changing character of the heat transfer dependence on the port-to-exhaust nozzle throat area ratio evidenced in Figs. 8–10 of the paper. Since no detailed measurements were made in the jet itself, it is not possible to ascertain the flow regime prevailing for each test.

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